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Existence and iteration of positive solutions for multi-point boundary value problems on a half-line

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ABSTRACT

In this study, using fixed point index theorems and monotone iterative techniques, we present iterative schemes as well as the existence of positive solutions to the m-point boundary value problem on a half-line.

1. Introduction

Consider the following multi-point boundary value problem in this study

$$\begin{cases} (\varphi_p(u'(t)))' + h(t)f(t, u(t)) = 0, \quad \text{a.e. } t \in (0, \infty), \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u'(\infty) = c_\infty \geq 0, \end{cases} \tag{P}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0, \infty)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty$, $a_i \in [0, 1)$ with $0 \leq \sum_{i=1}^{m-2} a_i < 1$, h is a nonnegative measurable function on $(0, \infty)$, and $f \in C([0, \infty) \times (0, \infty), \mathbb{R})$. Here, h may be singular at $t = 0$, and f may be singular at $u = 0$. Throughout the paper, let us assume the following assumption for weight function h.

(H) The weight function h is in $\mathcal{A}$, where

$$\mathcal{A} = \left\{ \bar{h} \in L^1_{\text{loc}}(0, \infty) \mid \int_0^\infty \varphi_p^{-1}\left(\int_s^\infty \bar{h}(\tau)\mathrm{d}\tau \right) \mathrm{d}s < \infty \right\}.$$

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initially done by Il'in and Moiseev [1,2]. Since then, many researchers have studied nonlinear second-order multi-point boundary value problems under various conditions of nonlinearity. In particular, there have been many papers concerned with the existence of one or multiple positive solutions to boundary value problems on the half-line, which arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium (see [3–11]).

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Most of these papers only considered the existence of positive solutions under certain conditions. Recently, for semilinear case ($p = 2$), Zhang [11] has not only investigated the existence of positive solutions for problem (P), but also established iterative schemes for approximating the solutions, in which it is considered simply the case that nonlinearity $f(t, u)$ is nonnegative and for weight function h, $h(t)$, $th(t) \in L^1(0, \infty)$. The first aim of this paper is to extend the result in [11] by assuming the weaker hypotheses to the weight function h than those in [11]. Note that if $h \in \mathcal{A}$, then $h \in L^1(a, \infty)$ for all $a > 0$, but $h \notin L^1(0, \infty)$. Moreover, we give the existence of a positive solution to problem (P) with sign-changing nonlinearity, which is new even for semilinear case.

For the case $c_\infty = 0$, there have been many papers, even for generalized-Laplacian case (see [5–9]). To the author's knowledge, in p-Laplacian case, there is no result for problem (P) with $c_\infty > 0$. The second aim of this paper is to fill the gap in this direction.

The rest of this paper is organized as follows. In Section 2, the corresponding operator to problem (P) is introduced, and well-known facts and lemmas are presented. In Section 3, we give main results such as iterative schemes and the existence of positive solutions to problem (P). Finally, examples to illustrate our results are given in Section 4.

2. Preliminaries

The following theorem is essential to the proofs of the main results in this paper.

Theorem 2.1 ([12]). *Let X be a Banach space, $\mathcal{K}$ a cone in X and $\mathcal{O}$ bounded and open in X. Let $0 \in \mathcal{O}$ and $T : \mathcal{K} \cap \overline{\mathcal{O}} \to \mathcal{K}$ be completely continuous such that $Tx \neq x$ for all $x \in \mathcal{K} \cap \partial\mathcal{O}$. Then the following results hold.*

(i) *If $\|Tx\| \le \|x\|$, then $i(T, \mathcal{K} \cap \mathcal{O}, \mathcal{K}) = 1$.*
(ii) *If $\|Tx\| \ge \|x\|$, then $i(T, \mathcal{K} \cap \mathcal{O}, \mathcal{K}) = 0$.*

Let $X = \left\{u \in C[0, \infty) \mid \sup_{0 \le t < \infty} \frac{|u(t)|}{1+t} < \infty\right\}$. Then X is a Banach space with norm $\|u\| = \sup_{0 \le t < \infty} \frac{|u(t)|}{1+t}$. Put $\mathcal{K} = \{u \in X | u$ is a nonnegative, nondecreasing, and concave function on $[0, \infty]$ satisfying $u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)\}$. Then, $\mathcal{K}$ is a cone in X.

By a positive solution of problem (P), we mean a function $u \in X \cap C^1(0, \infty)$ satisfies (P) and $u > 0$ in $(0, \infty)$.

Throughout the paper, let us assume the following assumption for nonlinearity f unless otherwise stated.

(F) $f \in C([0, \infty) \times [0, \infty), [0, \infty))$ and for each $w > 0$, there exists $M_w > 0$ such that $f(t, (1 + t)v) \le M_w$ for $(t, v) \in [0, \infty) \times [0, w]$.

Define $T : \mathcal{K} \to X$ by

$$(Tu)(t) = C(u) + \int_0^t K(u)(s)\mathrm{d}s, \quad 0 \le t < \infty,$$

where

$$C(u) = A^{-1} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1} \left[\varphi_p(c_\infty) + \int_s^\infty h(\tau)f(\tau, u(\tau))\mathrm{d}\tau\right] \mathrm{d}s,$$

$$K(u)(s) = \varphi_p^{-1} \left[\varphi_p(c_\infty) + \int_s^\infty h(\tau)f(\tau, u(\tau))\mathrm{d}\tau\right],$$

and

$$A = 1 - \sum_{i=1}^{m-2} a_i.$$

Since (F) and (H) are assumed, T is well defined and $Tu \in \mathcal{K}$ for all $u \in \mathcal{K}$. Furthermore, we can easily know that problem (P) has a positive solution u if and only if T has a fixed point u in $\mathcal{K} \setminus \{0\}$.

The following lemma is a well-known fact which will be used often in this paper.

Lemma 2.2 ([13, p. 73]). *For $q \in (0, \infty)$, put $d_q = \max\{1, 2^{q-1}\}$. Then*

$$|\alpha - \beta|^q \le d_q(|\alpha|^q + |\beta|^q)$$

for arbitrary complex numbers α and β.

Note that for any bounded subset Σ of $\mathcal{K}$, $C(u)$ is uniformly bounded on Σ. In fact, by the condition (F), there exists $N > 0$ such that $f(t, u(t)) < N$ for $u \in \Sigma$, $t \in [0, \infty)$. Thus,

$$|C(u)| \le A^{-1}\xi_{m-2} d_{\frac{1}{p-1}} \left[c_\infty + \varphi_p^{-1}(N) \int_0^\infty \varphi_p^{-1}\left(\int_s^\infty h(\tau)\mathrm{d}\tau\right) \mathrm{d}s\right] < \infty.$$

Here, $d_{\frac{1}{p-1}}$ is the constant in Lemma 2.2 with $q = 1/(p-1)$.

Remark 2.3. Assume (F) and (H). Then it is easy to see that if u is a positive solution of (P), then it is bounded if $c_\infty = 0$ and unbounded if $c_\infty > 0$.

To show the complete continuity of T, we use the following lemma.

Lemma 2.4 ([5]). *Let W be a bounded subset of $\mathcal{K}$. Then W is relatively compact in X if $\{W(t)/(1+t)\}$ are equicontinuous on any finite subinterval of $[0,\infty)$ and for any $\epsilon > 0$ there exists $N > 0$ such that*

$$\left|\frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2}\right| < \epsilon$$

uniformly with respect to $x \in W$ as $t_1, t_2 \geq N$, where $W(t) = \{x(t) : x \in W\},\ t \in [0,\infty)$.

Lemma 2.5. *T is completely continuous on $\mathcal{K}$.*

Proof. We first show that T is compact. Let Σ be bounded in $\mathcal{K}$, i.e., there exists $M > 0$ such that $\|u\| \leq M$ for all $u \in \Sigma$. By (F), there exists $N_1 > 0$ such that $f(t, u(t)) \leq N_1$ for all $t \in [0, 1],\ u \in \Sigma$. Then, we can easily show that $T(\Sigma)$ is bounded. Indeed,

$$\begin{aligned}\left|\frac{(Tu)(t)}{1+t}\right| &\leq A^{-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} d_{\frac{1}{p-1}}\left[c_\infty + \varphi_p^{-1}\left(N_1\int_s^\infty h(\tau)\right)\right]\mathrm{d}s + d_{\frac{1}{p-1}}\left[\frac{c_\infty t}{1+t} + \int_0^t \varphi_p^{-1}\left(N_1\int_s^\infty h(\tau)\mathrm{d}\tau\right)\right]\mathrm{d}s \\ &\leq A^{-1}d_{\frac{1}{p-1}}\left[c_\infty\xi_{m-2} + \varphi_p^{-1}(N_1)\int_0^\infty \varphi_p^{-1}\left(\int_s^\infty h(\tau)\mathrm{d}\tau\right)\right]\mathrm{d}s \\ &\quad + d_{\frac{1}{p-1}}\left[c_\infty + \varphi_p^{-1}(N_1)\int_0^\infty \varphi_p^{-1}\left(\int_s^\infty h(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right] < \infty.\end{aligned}$$

Thus, $T(\Sigma)$ is bounded.

For any $R > 0$ and $t_1, t_2 \in [0, R]$ with $t_1 < t_2$, we have

$$\begin{aligned}\left|\frac{Tu(t_1)}{1+t_1} - \frac{Tu(t_2)}{1+t_2}\right| &= \left|\frac{C(u) + \int_0^{t_1} K(u)(s)\mathrm{d}s}{1+t_1} - \frac{C(u) + \int_0^{t_2} K(u)(s)\mathrm{d}s}{1+t_2}\right| \\ &\leq C(u)\left|\frac{1}{1+t_1} - \frac{1}{1+t_2}\right| + \left|\frac{(1+t_2)\int_0^{t_1} K(u)(s)\mathrm{d}s - (1+t_1)\int_0^{t_2} K(u)(s)\mathrm{d}s}{(1+t_1)(1+t_2)}\right| \\ &\leq C(u)|t_1 - t_2| + (1+t_1)\int_{t_1}^{t_2} K(u)(s)\mathrm{d}s + (t_2 - t_1)\int_0^{t_1} K(u)(s)\mathrm{d}s \\ &\leq C(u)|t_1 - t_2| + (1+R)\int_{t_1}^{t_2} K(u)(s)\mathrm{d}s + (t_2 - t_1)\int_0^{R} K(u)(s)\mathrm{d}s,\end{aligned}$$

which yields, by the conditions (F) and (H), that $T\Sigma$ is equicontinuous on any finite subinterval of $[0,\infty)$.

For $u \in \Sigma$, by L'Hospital's rule, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{Tu(t)}{1+t} &= \lim_{t\to\infty}\frac{C(u) + \int_0^t K(u)(s)\mathrm{d}s}{1+t} \\ &= \varphi_p^{-1}\left[\varphi_p(c_\infty) + \lim_{t\to\infty}\int_t^\infty h(\tau)f(\tau, u)\mathrm{d}\tau\right].\end{aligned}$$

Note that since $h \in \mathcal{A}$, $h \in L^1(a,\infty)$ for all $a > 0$. It follows from the conditions (F) and (H) that $\frac{Tu(t)}{1+t} \to c_\infty$ as $t \to \infty$, uniformly on Σ. Thus, we can easily show that for any $\epsilon > 0$, there exists sufficiently large $L_0 > 0$ such that

$$\left|\frac{Tu(t_1)}{1+t_1} - \frac{Tu(t_2)}{1+t_2}\right| < \epsilon, \quad \text{for all } t_1, t_2 \geq L_0,\ u \in \Sigma.$$

By Lemma 2.4, we can conclude that T is compact on $\mathcal{K}$.

We finally show that T is continuous. Let $\{u_n\}$ be a sequence with u_n converges to u_0 in $\mathcal{K}$. Note that for any $t \in [0,\infty)$, $u_n(t) \to u_0(t)$ as $n \to \infty$. Since $\{u_n\}$ is bounded in $\mathcal{K}$, there exists $N_2 > 0$ such that $f(t, u_n(t)) \leq N_2$ for all $t \in [0,\infty)$, and by the compactness of T, there exists a subsequence, say again, $\{u_n\}$ such that Tu_n converges to V in X. Since it follows from Lebesgue dominated convergence theorem that $Tu_n(t) \to Tu_0(t),\ t \in [0,\infty)$, we have $V \equiv Tu_0$. So far we have shown that if a sequence $\{u_n\}$ converges to u_0 in $\mathcal{K}$, then there exists a subsequence, say $\{u_{n_j}\}$ such that

$$Tu_{n_j} \to Tu_0 \quad \text{in } X.$$

By the standard argument, we can easily show that the original sequence also satisfies

$$Tu_n \to Tu_0 \quad \text{in } X$$

and this completes the proof. □

Lemma 2.6. *For all $u \in \mathcal{K}$, $u(t) \geq \min\{t, 1\}\|u\|$ for $t \in [0, \infty)$.*

Proof. Let $\sigma = \inf\left\{\xi \in [0, \infty] : \|u\| = \lim_{t\to\xi} \frac{|u(t)|}{1+t}\right\}$. Note that σ may be ∞ if $c_\infty > 0$. We have two cases: either (i) $t < \sigma$ or (ii) $t \geq \sigma$. First, let us assume that $t < \sigma$. Then, by concavity of u, we have

$$\frac{u(t) - u(0)}{t} \geq \frac{u(s) - u(0)}{s}, \quad t < s < \sigma,$$

i.e.

$$\frac{u(t)}{t} \geq \frac{u(s)}{s} - \frac{u(0)}{s} + \frac{u(0)}{t} \geq \frac{u(s)}{1+s}.$$

Thus, letting $s \to \sigma$, we have $u(t) \geq t\|u\|$. For $t \geq \sigma$, since u is nondecreasing, we have

$$u(t) \geq u(\sigma) = (1 + \sigma)\|u\| \geq \|u\|.$$

Thus the proof is complete. □

For convenience, we use the following notations.

$$\mathcal{K}_r = \{u \in \mathcal{K} \mid \|u\| < r\},$$
$$\partial\mathcal{K}_r = \{u \in \mathcal{K} \mid \|u\| = r\},$$
$$\Omega_r = \left\{u \in \mathcal{K} \mid \min_{t\in[1/k,k]} \frac{u(t)}{1+t} < \gamma_k r\right\},$$
$$f_{\gamma_k R,R} = \min\left\{\frac{f(t, (1+t)v)}{\varphi_p(R)} \mid t \in [k^{-1}, k],\ v \in [\gamma_k R, R]\right\},$$
$$f^{0,r} = \sup\left\{\frac{f(t, (1+t)v)}{\varphi_p(r)} \mid t \in [0, \infty),\ v \in [0, r]\right\},$$
$$M = \left(A^{-1}\sum_{i=1}^{m-2} a_i\xi_i + 1\right) d_{\frac{1}{p-1}} c_\infty,$$
$$N = \left(2d_{\frac{1}{p-1}}\left[A^{-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1}\left(\int_s^\infty h(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \int_0^\infty \varphi_p^{-1}\left(\int_s^\infty h(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\right)^{-1},$$
$$L = (1+k)\left[\int_0^{1/k} \varphi_p^{-1}\left(\int_{1/k}^k h(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]^{-1},$$

where k is a fixed constant satisfying $0 < 1/k < \xi_1 < \xi_{m-2} < k < \infty$ and $\gamma_k = [k(k+1)]^{-1}$.

Remark 2.7. By Lemma 2.6, one can easily see the following facts.

(1) For any $u \in \mathcal{K}$,

$$\gamma_k\|u\| \leq \frac{u(t)}{1+t}, \quad t \in \left[\frac{1}{k}, k\right].$$

(2) By (1), one has

$$\Omega_r = \left\{u \in \mathcal{K} \mid \gamma_k\|u\| \leq \min_{t\in[1/k,k]} \frac{u(t)}{1+t} < \gamma_k r\right\}.$$

Lemma 2.8 ([14])**.** *Ω_r has the following properties.*

(1) Ω_r is open relative to $\mathcal{K}$.
(2) $\mathcal{K}_{\gamma_k r} \subseteq \Omega_r \subseteq \mathcal{K}_r$.
(3) $u \in \partial\Omega_r$ if and only if $\min_{t\in[1/k,k]} \frac{u(t)}{1+t} = \gamma_k r$.
(4) If $u \in \partial\Omega_r$, then $\gamma_k r \leq \frac{u(t)}{1+t} \leq r$ for $t \in [1/k, k]$.

Lemma 2.9. *Assume that there exists $r > 0$ such that*

(H_1^r) $r \geq 2M$ and $f^{0,r} \leq \varphi_p(N)$,

then $\|Tu\| \leq \|u\|$ for $u \in \partial\mathcal{K}_r$. Furthermore, if

$(H_1^r)^*$ $r \geq 2M$ and $f^{0,r} < \varphi_p(N)$

is assumed instead of (H_1^r), then $i(T, \mathcal{K}_r, \mathcal{K}) = 1$.

Proof. Assume (H_1^r). For $u \in \partial\mathcal{K}_r$, we have $u(t) \leq (1+t)r$ and $f(t, u(t)) \leq \varphi_p(Nr)$, $t \in [0, \infty)$. Then,

$$\begin{aligned}\left|\frac{Tu(t)}{1+t}\right| &\leq A^{-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} d_{\frac{1}{p-1}}\left[c_\infty + Nr\varphi_p^{-1}\left(\int_s^\infty h(\tau)\mathrm{d}\tau\right)\right]\mathrm{d}s + \frac{\int_0^t d_{\frac{1}{p-1}}\left[c_\infty + Nr\varphi_p^{-1}\left(\int_s^\infty h(\tau)\mathrm{d}\tau\right)\right]\mathrm{d}s}{1+t} \\ &= \left(A^{-1}\sum_{i=1}^{m-2} a_i\xi_i + 1\right) d_{\frac{1}{p-1}} c_\infty + d_{\frac{1}{p-1}} Nr\left[A^{-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1}\left(\int_s^\infty h(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right. \\ &\quad \left. + \int_0^\infty \varphi_p^{-1}\left(\int_s^\infty h(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right] \\ &\leq r = \|u\|.\end{aligned}$$

Assume $(H_1^r)^*$. Then, $\|Tu\| < \|u\|$ for $u \in \partial\mathcal{K}_r$ by similar calculation, and thus $i(T, \mathcal{K}_r, \mathcal{K}) = 1$ in view of (i) of Theorem 2.1. □

Lemma 2.10. *Assume that there exists $R > 0$ such that*

(H_2^R) $f_{\gamma_k R, R} \geq \varphi_p(L)$,

then $\|Tu\| \geq \|u\|$ for $u \in \Omega_R$. Furthermore, if

$(H_2^R)^*$ $f_{\gamma_k R, R} > \varphi_p(L)$

is assumed instead of (H_2^R), then $i(T, \Omega_R, \mathcal{K}) = 0$.

Proof. Assume (H_2^R). Then for $u \in \partial\Omega_R$, we have

$$\gamma_k R \leq \frac{u(t)}{1+t} \leq R, \quad t \in \left[\frac{1}{k}, k\right],$$

and $\|u\| \leq R$ by Lemma 2.8. It follows from (H_2^R) that

$$f(t, u(t)) \geq \varphi_p(LR), \quad t \in [1/k, k].$$

This implies, for $u \in \partial\Omega_R$ and $t \in [1/k, k]$,

$$\begin{aligned}\|Tu\| &\geq \frac{Tu(t)}{1+t} \\ &\geq \frac{\int_0^t \varphi_p^{-1}\left[\int_s^\infty h(\tau)f(\tau, u)\mathrm{d}\tau\right]\mathrm{d}s}{1+t} \\ &\geq \frac{RL\left[\int_0^{\frac{1}{k}} \varphi_p^{-1}\left(\int_{1/k}^k h(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]}{1+k} \\ &\geq R \geq \|u\|.\end{aligned}$$

Assume $(H_2^R)^*$. Then, it follows that $\|Tu\| > \|u\|$ for $u \in \partial\Omega_R$. Thus $i(T, \Omega_R, \mathcal{K}) = 0$ in view of (ii) of Theorem 2.1. □

3. Main results

Now we give our results for the existence of positive solutions of problem (P).

Theorem 3.1. *Assume that there exist constants $r, R > 0$ with $0 < R < r$ (or $0 < r < \gamma_k R$) such that conditions (H_1^r) and (H_2^R) hold. Then problem (P) has a positive solution u such that $\gamma_k R \leq \|u\| \leq r$ (or $r \leq \|u\| \leq R$), respectively.*

Proof. We only prove the case $R < r$ since the other case is similar. If there exists $u \in \partial\mathcal{K}_r \cup \partial\Omega_R$ such that $Tu = u$, the proof is done. Otherwise, by Theorem 2.1, Lemmas 2.9 and 2.10, $i(T, \mathcal{K}_r, \mathcal{K}) = 1$ and $i(T, \Omega_R, \mathcal{K}) = 0$, and it follows from the additivity property that $i(T, \mathcal{K}_r \setminus \overline{\Omega}_R, \mathcal{K}) = -1$. Then there exists $u \in \mathcal{K}_r \setminus \overline{\Omega}_R$ such that $Tu = u$ by the solution property. Thus, the proof is complete in view of (2) of Lemma 2.8. □

The following corollary directly follows from Lemmas 2.9 and 2.10.

Corollary 3.2. *Assume that there exist constants $r, R > 0$ with $0 < R < r$ (or $0 < r < \gamma_k R$) such that conditions $(H_1^r)^*$ and $(H_2^R)^*$ hold. Then problem (P) has a positive solution u such that $\gamma_k R \le \|u\| < r$ (or $r < \|u\| \le R$), respectively.*

Remark 3.3. One can easily obtain the result that problem (P) has arbitrarily many positive solutions by combining conditions $(H_1^{r_i})^*$, $(H_2^{R_i})^*$ $(i = 1, 2, \ldots)$ properly (e.g., see Theorem 2.11 in [14]).

Now we give the monotone iterative schemes for approximating a positive solution to problem (P).

Theorem 3.4. *Assume that there exists $r > 0$ such that the condition (H_1^r) holds. Assume, in addition,*

(F_1) *$f(t, (1+t)v_1) \le f(t, (1+t)v_2)$ for $t \in [0, \infty)$, $0 \le v_1 \le v_2 \le r$,*
(F_2) *either $f(t, 0) \not\equiv 0$ for $t \in [0, \infty)$ or $c_\infty > 0$.*

Then problem (P) has a positive solution z^* such that $0 < \|z^*\| \le r$, and $\lim_{n\to\infty} z_n = \lim_{n\to\infty} T^n z_0 = z^*$, where $z_0 \equiv 0$.

Proof. Let $z_0(t) = 0$, $t \in [0, \infty)$ and $z_n = Tz_{n-1} (n = 1, 2, \ldots)$. Then, by the same argument as in the proof of Lemma 2.9, $\|z_n\| \le r$ for all n. It follows from the compactness of T that $\{z_n\}$ is a sequentially compact set. Clearly, $z_1 \ge 0 \equiv z_0$. By (F_1) and induction, we get $z_n \ge z_{n-1}$ $(n = 1, 2, \ldots)$, which implies $z_n \to z^*$ in $\mathcal{K}$ and $\|z^*\| \le r$. It follows from the continuity of T that $Tz^* = z^*$, and thus z^* is a positive solution of (P) in view of (F_2). This completes the proof. □

Remark 3.5. Note that the condition (F_1) is equivalent to the condition

(F_1') $f(t, u_1) \le f(t, u_2)$ for $0 \le u_1 \le u_2 \le r(1+t)$, $t \in [0, \infty)$.

Thus if (F_1) is assumed, one can easily see that T is nondecreasing for $u \in \overline{\mathcal{K}_r}$, i.e. $Tu_1 \le Tu_2$ for any $u_1, u_2 \in \overline{\mathcal{K}_r}$ with $u_1 \le u_2$.

Theorem 3.6. *Assume that $f \in C([0, \infty) \times (0, \infty), \mathbb{R})$ and there exist $r, R > 0$ such that $0 < R < r$, $r \ge 2M$ and the condition (H_2^R) holds. Assume, in addition,*

(F_3) *$0 \le f(t, (1+t)v) \le \varphi_p(Nr)$ for $(t, v) \in ([0, 1/k] \times [0, r]) \cup ([1/k, k] \times [\gamma_k R, r]) \cup ([k, \infty) \times [0, r])$,*
(F_4) *$f(t, (1+t)v_1) \le f(t, (1+t)v_2)$ for $(t, v_1), (t, v_2) \in ([0, 1/k] \times [0, r]) \cup ([1/k, k] \times [\gamma_k R, r]) \cup ([k, \infty) \times [0, r])$ and $v_1 \le v_2$.*

Then problem (P) has a positive solution w^* such that $R \le \|w^*\| \le r$, and $\lim_{n\to\infty} w_n = \lim_{n\to\infty} \overline{T}^n w_0 = w^*$, where $w_0(t) = r(1+t)$, $t \in [0, \infty)$ and $\overline{T}$ is the corresponding operator to the modified problem $(\overline{\mathrm{P}})$ below.

Proof. Consider the modified problem

$$\begin{cases} (\varphi_p(u'(t)))' + h(t)\overline{f}(t, u) = 0, \quad \text{a.e. } t \in (0, \infty), \\ u(0) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i), \quad u'(\infty) = c_\infty, \end{cases} \tag{$\overline{\mathrm{P}}$}$$

where $\overline{f}$ is defined by

$$\overline{f}(t, u) = \begin{cases} f(t, r(1+t)) \quad \text{for } (t, u) \in [0, \infty) \times [r(1+t), \infty), \\ f(t, u) \quad \text{for } (t, u) \in (([0, 1/k] \cup [k, \infty)) \times [0, r(1+t))) \cup ((1/k, k) \times [\gamma_k R(1+t), r(1+t))), \\ \dfrac{f(k, u) - f(1/k, u)}{(k - 1/k)}(t - 1/k) + f(1/k, u) \quad \text{for } (t, u) \in (1/k, k) \times [0, \gamma_k R(1+t)). \end{cases}$$

By (F_3) and (F_4), $\overline{f}$ satisfies (F) and (F_1), and $\overline{T} : \mathcal{K} \to \mathcal{K}$ is completely continuous and nondecreasing for $u \in \overline{\mathcal{K}_r}$. Moreover, it follows from (F_3) and (H_2^R) that $\overline{f}^{0,r} \le \varphi_p(N)$ and $\overline{f}_{\gamma_k R, R} \ge \varphi_p(L)$. From these facts, if $R \le \|u\| \le r$, then we have

$$\overline{f}(t, u(t)) \le \overline{f}(t, r(1+t)) \le \varphi_p(Nr), \quad t \in [0, \infty)$$

and

$$\overline{f}(t, u(t)) \ge \overline{f}(t, \gamma_k R(1+t)) \ge \varphi_p(LR), \quad t \in [1/k, k],$$

which yield $R \le \|\overline{T}u\| \le r$ by the similar arguments as in the proofs of Lemmas 2.9 and 2.10.

Let $w_0(t) = r(1+t)$ for $t \in [0, \infty)$, and $w_n = \overline{T} w_{n-1}$ $(n = 1, 2, \ldots)$. Then, $R \le \|w_n\| \le r$ $(n = 0, 1, 2, \ldots)$. It follows from the compactness of $\overline{T}$ that $\{w_n\}$ is a sequentially compact set. Since $\|w_1\| \le r$, we have $w_1(t) \le r(1+t) = w_0(t),\ t \in [0, \infty)$. By induction, we get $w_n \le w_{n-1}$ $(n = 1, 2, \ldots)$. By the standard argument, we can conclude that there exists a positive solution w^* of $(\overline{P})$ such that $R \le \|w^*\| \le r$, and $\lim_{n\to\infty} w_n = \lim_{n\to\infty} \overline{T}^n w_0 = w^*$. Since $R \le \|w^*\| \le r$, we have $\overline{f}(t, w^*(t)) = f(t, w^*(t)),\ t \in [0, \infty)$. Thus w^* is a positive solution of (P), and this completes the proof. □

Remark 3.7. The iterative schemes in Theorems 3.4 and 3.6 start off with the known zero function and simple linear function, respectively. Thus the iterative schemes are convenient and feasible.

4. Examples

In this section, we give examples to illustrate our results obtained in Section 3. Consider the following three-point boundary value problem

$$\begin{cases} (|u'(t)|u'(t))' + h(t)f(t, u(t)) = 0, & t \in (0, \infty), \\ u(0) = \dfrac{1}{2}u(1), \quad u'(\infty) = 1, \end{cases} \tag{4.1}$$

where

$$h(t) = \begin{cases} t^{-4}, & t \ge 1, \\ t^{-2}, & 0 < t \le 1. \end{cases}$$

Note that $h \in \mathcal{A}$ for $p = 3$, but $h \notin L^1(0, \infty)$. Choose $k = 2$. One can easily know that $\gamma_k = 1/6, M = 2, N \ge 1/12$, and $L < 6$.

(1) Let us define f by

$$f(t, u) = \frac{1}{2}|\sin t| + \frac{1}{2^9}\left(\frac{u}{1+t}\right)^2, \quad (t, u) \in [0, \infty) \times [0, \infty).$$

Choose $r = 2^4$. Then by direct calculation one can know that all conditions of Theorem 3.4 are satisfied. Thus problem (4.1) has a positive solution z^* such that $0 < \|z^*\| \le 2^4$, and $\lim_{n\to\infty} z_n = \lim_{n\to\infty} T^n z_0 = z^*$, where $z_0 \equiv 0$.

(2) Let us define f by

$$f(t, u) = \begin{cases} 36, & (t, u) \in \left(\left[0, \dfrac{1}{2}\right] \cup [2, \infty)\right) \times \left[0, \dfrac{1+t}{6}\right], \\ g(t, u), & (t, u) \in \left(\dfrac{1}{2}, 2\right) \times \left[0, \dfrac{1+t}{6}\right], \\ \dfrac{u}{1+t} - \dfrac{1}{6} + 36, & (t, u) \in [0, \infty) \times \left(\dfrac{1+t}{6}, 1+t\right], \\ \dfrac{5}{6} + 36, & (t, u) \in [0, \infty) \times (1+t, \infty), \end{cases}$$

where g is defined as an appropriate way which enables f to be continuous. Note that g may have negative values and be singular at $u = 0$. Choose $R = 1$ and $r = 112$. Clearly, (F_4) holds. For $v \in [1/6, 1], f(t, (1+t)v) = v - 1/6 + 36 \ge 36 \ge \varphi_3(L) = \varphi_3(LR)$, which implies (H_2^R) holds. Finally, (F_3) holds since $f(t, (1+t)v) \le 5/6 + 36 \le \varphi_3(112N) = \varphi_3(Nr)$ for $(t, v) \in ([0, 1/2] \times [0, 112]) \cup ([1/2, 2] \times [1/6, 112]) \cup ([2, \infty) \times [0, 112])$. Thus all conditions of Theorem 3.6 are satisfied, and problem (4.1) has a positive solution w^* such that $1 \le \|w^*\| \le 112$ and $\lim_{n\to\infty} w_n = \lim_{n\to\infty} \overline{T}^n w_0 = w^*$, where $w_0(t) = 112(1+t),\ t \in [0, \infty)$.

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